The Mean Field Approach to Sherrington-Kirkpatrick Model

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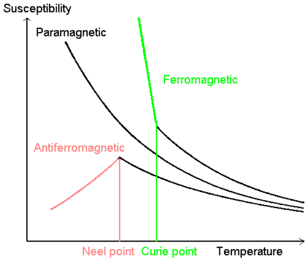
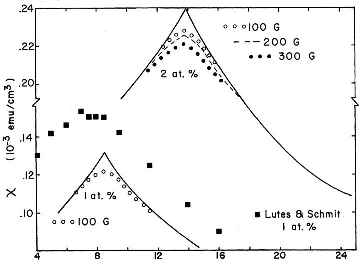
A Little History

- **Experimental finding:** In some dilute **magnetic alloys** such as AuFe and CuMn, when the temperature drops, the magnetization increases slowly, and then decreases slowly after a peak.





the anti-ferromagnetic material: oxides, halides, sulfides, etc.

- the first name: **mictomagnet**, coming from the mix of ferromagnetic and antiferromagnetic interactions.
- In 1970, P. Anderson proposed the concept of **spin glass** at the suggestion of B. R. Coles.
  Anderson P W. Localisation theory and the CuMn problem: Spin glasses. Materials Research Bulletin, 1970, 5(8): 549-554.

A Little History

- the first model: **Edwards-Anderson model** (Edwards & Anderson, 1975)

$$H = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j \tag{1}$$

Edwards S F, Anderson P W. Theory of spin glasses. Journal of Physics F: Metal Physics, 1975, 5(5): 965.

- the most famous model: **Sherrington-Kirkpatrick model** (Sherrington & Kirkpatrick, 1975)

$$H = -\sum_{i<j} J_{ij} S_i S_j \tag{2}$$

Sherrington D, Kirkpatrick S. Solvable model of a spin-glass. Physical Review Letters, 1975, 35(26): 1792.

- **the replica symmetry solution** (Sherrington & Kirkpatrick, 1975)
  The order parameters m and q are the fixed points of the following equations

$$m = \int Dz \tanh \beta \widetilde{H}(z) \tag{3}$$

$$q = \int Dz \tanh^2 \beta \widetilde{H}(z) \tag{4}$$

where $Dz \equiv \mathrm{d}z\, e^{-z^2/2}/\sqrt{2\pi}$ and $\widetilde{H}(z) \equiv J\sqrt{q}z + J_0 m + h$. The free energy is calculated as

$$f_{\mathrm{RS}} = -\frac{1}{4}\beta J^2 (1-q)^2 + \frac{1}{2} J_0 m^2 - \frac{1}{\beta} \int Dz \log\left(2\cosh \beta \widetilde{H}(z)\right) \tag{5}$$

A Little History

- **the negative entropy crisis** (Sherrington & Kirkpatrick, 1975)
  When $h = 0$, $J_0 = 0$, and $T \to 0$, the free energy is

$$f_{\mathrm{RS}} = -\sqrt{\frac{2}{\pi}}J + \frac{T}{2\pi} \tag{6}$$

  $F = E - TS$, the ground state energy is $E = -\sqrt{2/\pi}J$, and the ground state entropy is $S = -\pi/2$.

- **the Thouless-Anderson-Palmer solution** (TAP equation, 1977)
  The local magnetization satisfies

$$m_i = \tanh \beta \left(\sum_j J_{ij} m_j + h - \beta \sum_j J_{ij}^2 (1 - m_j^2) m_i \right) \tag{7}$$

  The free energy is calculated as

$$f_{\mathrm{TAP}} = -\frac{1}{2} \sum_{i \neq j} J_{ij} m_i m_j - \sum_i h_i m_i - \frac{\beta}{4} \sum_{i \neq j} J_{ij}^2 (1 - m_i^2)(1 - m_j^2) + \frac{1}{2\beta} \sum_i \left\{ (1 + m_i) \log \frac{1 + m_i}{2} + (1 - m_i) \log \frac{1 - m_i}{2} \right\} \tag{8}$$

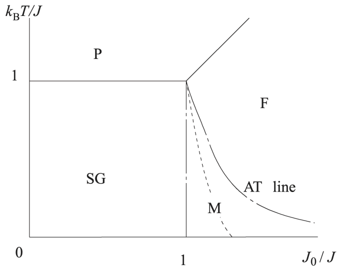
  Thouless D J, Anderson P W, Palmer R G. Solution of 'solvable model of a spin glass'. Philosophical Magazine, 1977, 35(3): 593-601.

A Little History

- the stability of the replica symmetric solution (dAT line, 1978)

$$\left(\frac{T}{J}\right)^2 > \int Dz\ \mathrm{sech}^4\left(\beta J\sqrt{q}z + \beta J_0 m\right) \text{ for } h = 0, \quad \left(\frac{T}{J}\right)^2 > \int Dz\ \mathrm{sech}^4\left(\beta J\sqrt{q}z + \beta h\right) \text{ for } J_0 = 0 \tag{9}$$

de Almeida J R L, Thouless D J. Stability of the Sherrington-Kirkpatrick solution of a spin glass model. Journal of Physics A: Mathematical and General, 1978, 11(5): 983.



A Little History

- **the replica symmetry breaking solution** (Parisi, 1979)
  The free energy of the RSB solution is given by

$$f_{\mathrm{RSB}} = -\frac{1}{4}\beta J^2 \left\{ 1 + \int_0^1 q(x)^2 \mathrm{d}x - 2q(1) \right\} - \frac{1}{\beta} \int Du\, f_0(0, \sqrt{q(0)}u) \tag{10}$$

  where f_0 satisfies the following Parisi equation:

$$\frac{\partial f_0(x,h)}{\partial x} = -\frac{J^2}{2} \frac{\mathrm{d}q}{\mathrm{d}x} \left\{ \frac{\partial^2 f_0}{\partial h^2} + x \left(\frac{\partial f_0}{\partial h^2} \right) \right\} \tag{11}$$

  Parisi G. Infinite number of order parameters for spin-glasses. Physical Review Letters, 1979, 43(23): 1754.
  Parisi G. A sequence of approximated solutions to the S-K model for spin glasses. Journal of Physics A: Mathematical and General, 1980, 13(3): 1101.

- Plefka expansion (1982)
  see SMNN chapter 5 "High-Temperature Expansion"
  Plefka T. Convergence condition for the TAP equation. Journal of Physics A: Mathematical and General, 1982, 15(6): 1971.

A Little History

- **the cavity method** (Mezard, Parisi, & Virasoro 1987)
  Mezard M, Parisi G, Virasoro M A. The Replica Solution without Replicas. Journal of Physics A: Mathematical and General, 1987, 20(9): 2555.
- The **vanilla cavity method** can directly derive the **TAP equations**, and through a more rough approximation, it can back to the **RS solution**.
- Later, this method was found to be powerful in dealing with optimization problems, such as the K-satisfiability problem.
  Martin O. C., Monasson R. and Zecchina R. Statistical mechanics methods and phase transitions in optimization problems. Theoretical Computer Science, 2002, 265(1-2): 3-67.
  Mezard M., Zecchina R. Random K-satisfiability problem: From an analytic solution to an efficient algorithm. Physical Review E, 2002, 66(5): 056126.
- It is equivalent to belief propagation, also known as sum-product message passing, which is an efficient algorithm applied to sparse graph models, as discussed in Chapter 2 in SMNN book.
  Yedidia J S, Freeman W T, Weiss Y. Constructing free-energy approximations and generalized belief propagation algorithms. IEEE Transactions on Information Theory, 2005, 51(7): 2282-2312.
  Nishimori H. Statistical Physics of Spin Glasses and Information Processing: An Introduction. Oxford University Press, 2001.
  Mezard M., Montanari A. Information, Physics, and Computation. Oxford University Press, 2009.

A Little History

- Although the SK model considers fully connected interactions, which is a dense graph rather than a sparse one, we can still do approximate calculation by the cavity iteration equation as following

$$h_{i\to ij} = \frac{1}{\beta}\Bigg(\sum_{k(\neq i,j)} \beta u_{ik\to i}\Bigg) \tag{12}$$

$$u_{ij\to i} = \frac{1}{\beta}\tanh^{-1}\Big[\tanh\big(\beta J_{ij}\big)\tanh\big(\beta h_{j\to ij}\big)\Big] \tag{13}$$

By iteratively solving for $u_{ij\to i}$, we can then solve for m_i using

$$m_i = \tanh\Bigg(\sum_{j(\neq i)} \beta u_{ij\to i}\Bigg), \tag{14}$$

and then we can calculate the free energy by $F = \sum_i \Delta F_i - \sum_{\langle ij\rangle} \Delta F_{ij}$, where

$$\Delta F_{ij} = \ln\Big[\cosh(\beta J_{ij})\Big(1 + \tanh(\beta J_{ij}) m_{i\to ij} m_{j\to ij}\Big)\Big] \tag{15}$$

$$\Delta F_i = \ln\Bigg(\prod_{j\neq i} \Lambda^{+}_{ij\to i} + \prod_{j\neq i} \Lambda^{-}_{ij\to i}\Bigg) \tag{16}$$

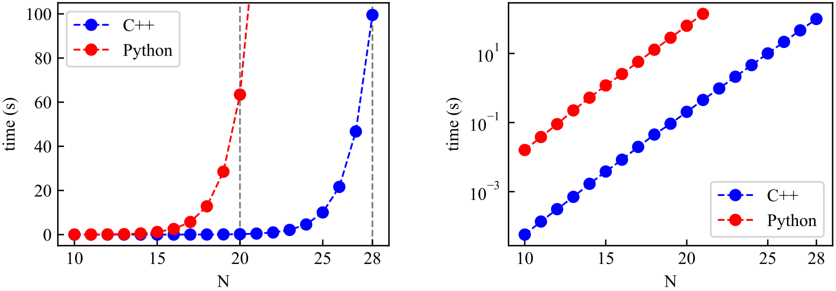
and

$$\Lambda^{\pm}_{ij\to i} = \cosh(\beta J_{ij})\Big[1 \pm \tanh(\beta J_{ij}) m_{ij\to i}\Big] \tag{17}$$

Equations (12)∼(17) can converge to the RS solution in the limit $N \to \infty$.

Some Tricks: the Enumeration Method

C++ or Python ?



Some Tricks: the TAP Equation

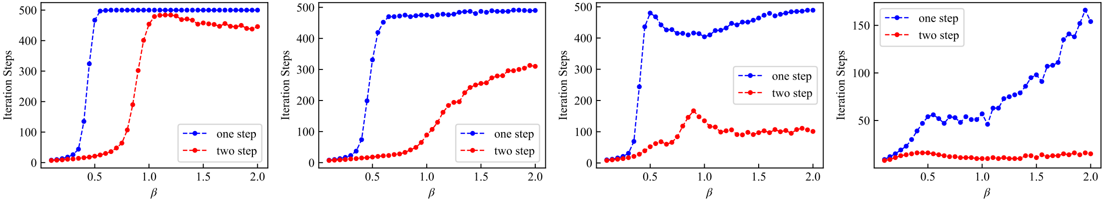
the one step iteration:

$$m_i^t = \tanh \beta \left(\sum_j J_{ij} m_j^{t-1} + h - \beta \sum_j J_{ij}^2 \left(1 - (m_j^{t-1})^2\right) m_i^{t-1} \right) \tag{18}$$

the two step iteration:

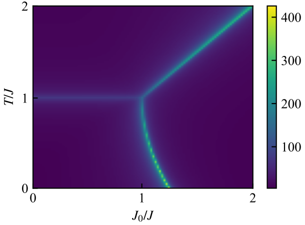
$$m_i = \tanh \beta \left(\sum_j J_{ij} m_j^{t-1} + h - \beta \sum_j J_{ij}^2 \left(1 - (m_j^{t-1})^2\right) m_i^{t-2} \right) \tag{19}$$

Erwin Bolthausen, An Iterative Construction of Solutions of the TAP Equations for the Sherrington-Kirkpatrick Model, Communications in Mathematical Physics 325, 333366 (2014).

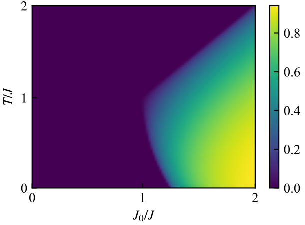


(a) $J_0 = 0, h = 0$ (b) $J_0 = 0.5, h = 0.5$ (c) $J_0 = 2, h = 0$ (d) $J_0 = 2, h = 0.5$

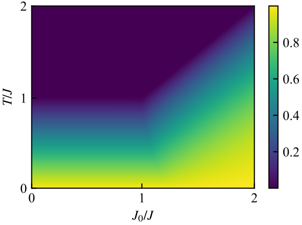
The Phase Diagrams



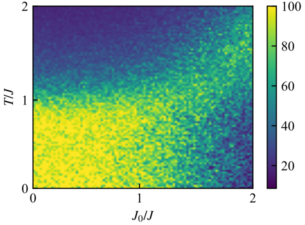
(e) iteration steps of RS solution



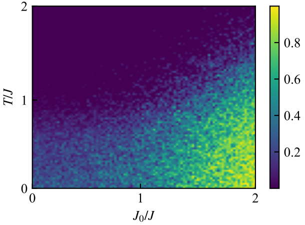
(f) magnetization m of RS solution



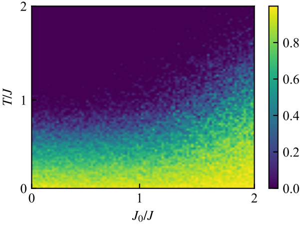
(g) EA order parameter q of RS solution



(h) iteration steps of cavity solution

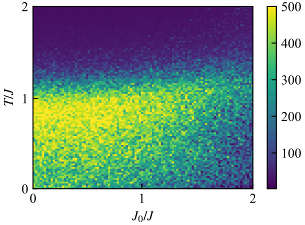


(i) magnetization m of cavity solution

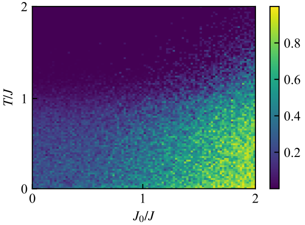


(j) EA order parameter q of cavity solution

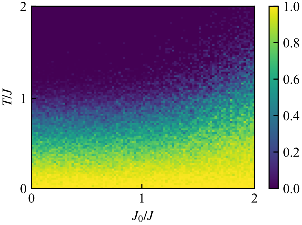
The Phase Diagrams



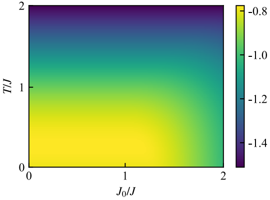
(k) iteration steps of TAP solution



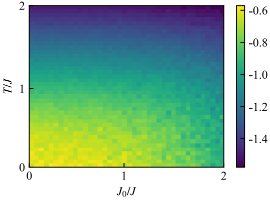
(l) magnetization m of TAP solution



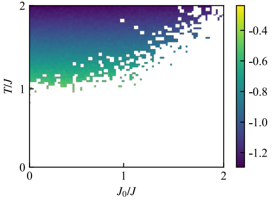
(m) EA order parameter q of TAP solution



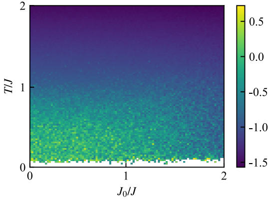
(n) free energy of RS solution



(o) free energy of enumeration



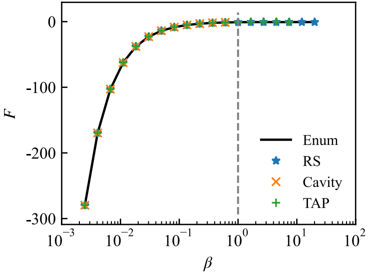
(p) free energy of cavity solution

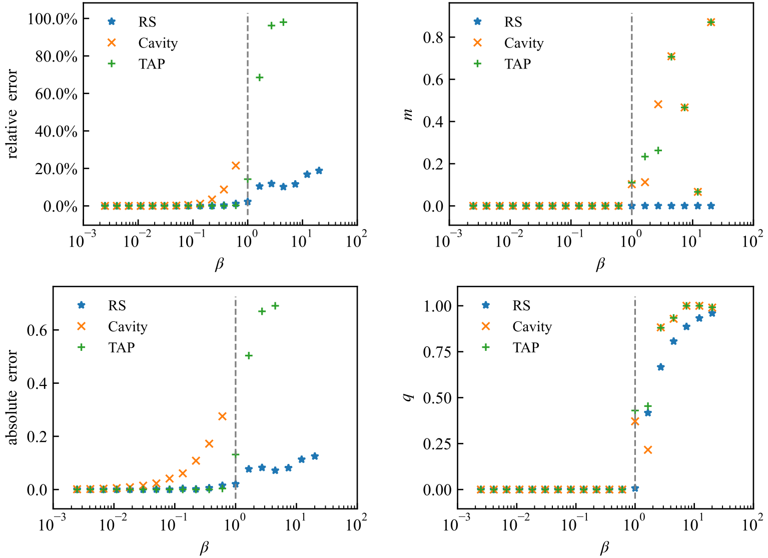


(q) free energy of TAP solution

Comparison of Different Methods

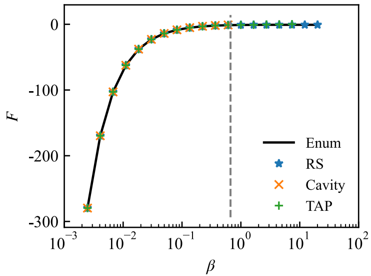
$J_0 = 0.5, J = 1, h = 0, N = 20$

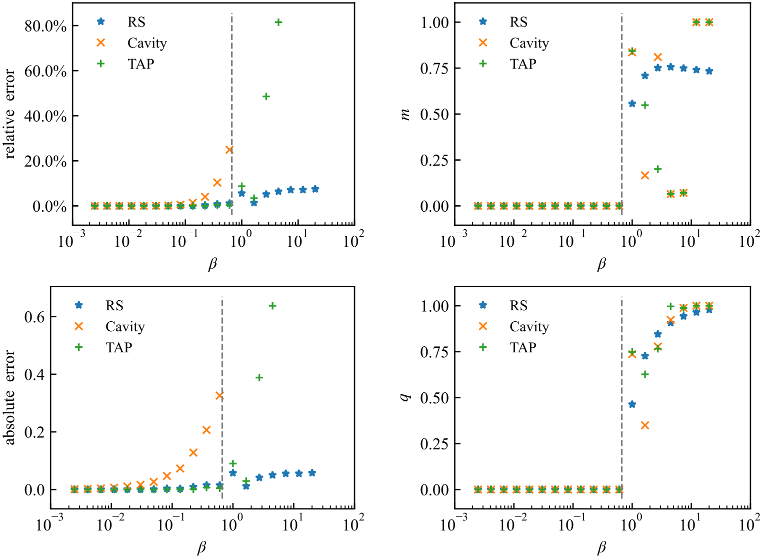




Comparison of Different Methods

$J_0 = 1.5$, $J = 1$, $h = 0$, $N = 20$





Comparison of Different Methods

$J_0 = 10, J = 1, h = 0, N = 20$