The Replica Symmetric Solution of Sherrington Kirkpatrick Model

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Abstract

This note reviews the replica symmetric solution of the SK model, including sufficiently detailed derivations, and then shows the phase diagrams of the order parameters and free energy by numerical calculations. The author acknowledges Prof. Hidetoshi Nishimori's book [1] for its introduction to the mean-field theory of spin glass, and Dr. Haozhe Shan's notes [2], which contain extensive derivations and greatly assisted the author in following the derivation.

1 Introduction

The Hamiltonian of the Sherrington-Kirkpatrick model reads [3]

$$H = -\sum_{i<j} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i, \tag{1}$$

where $\sigma_i \in \{1, -1\}$ are Ising spins, the interaction J_{ij} between any two spins is a quenched variable with the Gaussian distribution $J_{ij} \sim \mathcal{N}(J_0/N, J^2/N)$. The mean and variance are both proportional to $1/N$ to ensure the Hamiltonian is extensive. The probability of each configuration is given by the Gibbs-Boltzmann distribution $P(\boldsymbol{\sigma}) = \exp(-\beta H)/Z$, where Z is the partition function. We denote $\mathrm{Tr} = \sum_{\{\boldsymbol{\sigma}\}} = \sum_{\sigma_1=\pm 1}\cdots\sum_{\sigma_N=\pm 1}$, so that the partition function is expressed as $Z = \mathrm{Tr}\exp(-\beta H)$. The free energy F can be calculated by the partition function as $F = -\beta^{-1}\log Z$. However, it is only the free energy for a fixed interaction $\boldsymbol{J}$ sampled from the distribution. One not depended on any specific system sample can be obtain by the average over the distribution of $\boldsymbol{J}$, which is called the quenched average, or disorder average, or configurational average, and denoted by $\langle\cdot\rangle$ in this paper:

$$\langle F\rangle = -\frac{1}{\beta}\langle\log Z\rangle. \tag{2}$$

The dependence of $\langle\log Z\rangle$ on $\boldsymbol{J}$ is so complex that it cannot be solved directly, and this is where the replica method comes into play.

2 Replica Trick

The replica trick is a mathematical technique based on the application of the formula

$$\langle\log Z\rangle = \lim_{n\to 0}\frac{\langle Z^n\rangle - 1}{n}. \tag{3}$$

In this case, the replica average of the partition function can be written as

$$\langle Z^n\rangle = \int \mathcal{D}J \ \mathrm{Tr} \ \exp\left(\beta\sum_{i<j}J_{ij}\sum_{\alpha=1}^n S_i^\alpha S_j^\alpha + \beta h\sum_{i=1}^N\sum_{\alpha=1}^n S_i^\alpha\right), \tag{4}$$

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[1] Nishimori, Hidetoshi, *Statistical Physics of Spin Glasses and Information Processing: An Introduction* (Oxford, 2001), chap. 1~3

[2] Replica calculations for the SK model, URL: https://hzshan.github.io/replica_method_in_SK_model.pdf

[3] D. Sherrington and S. Kirkpatrick, *Solvable Model of a Spin-Glass*, Phys. Rev. Lett. **35**, 1792 (1975).

where the explicit expression of the integral measure $\mathcal{D}J$ is given by the distribution of J_{ij}

$$\mathcal{D}J = \prod_{i<j} \left[dJ_{ij} \, P\left(J_{ij}\right) \right] \propto \prod_{i<j} dJ_{ij} \, \exp\left[\sum_{i<j} -\frac{N}{2J^2} \left(J_{ij} - \frac{J_0}{N} \right)^2 \right].$$ (5)

Then, Eq. (4) is calculated as

$$\langle Z^n \rangle \propto \exp\left[\beta h \sum_{i=1}^{N} \sum_{\alpha=1}^{n} S_i^\alpha \right] \mathrm{Tr} \prod_{i<j} \left\{ \int dJ_{ij} \, \exp\left[-\frac{N}{2J^2} J_{ij}^2 + \left(\frac{J_0}{J^2} + \beta \sum_{\alpha=1}^{n} S_i^\alpha S_j^\alpha \right) J_{ij} \right] \right\}$$ (6a)

$$\propto \mathrm{Tr} \, \exp\left\{ \frac{1}{N} \sum_{i<j} \left(\frac{1}{2} \beta^2 J^2 \sum_{\alpha,\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + \beta J_0 \sum_{\alpha} S_i^\alpha S_j^\alpha \right) + \beta h \sum_{i=1}^{N} \sum_{\alpha=1}^{n} S_i^\alpha \right\}$$ (6b)

$$= \mathrm{Tr} \, \exp\left\{ \frac{1}{N} \sum_{i<j} \left(\frac{1}{2} \beta^2 J^2 \left(2 \sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + n \right) + \beta J_0 \sum_{\alpha} S_i^\alpha S_j^\alpha \right) + \beta h \sum_{i=1}^{N} \sum_{\alpha=1}^{n} S_i^\alpha \right\}$$ (6c)

$$= \mathrm{Tr} \, \exp\left\{ \frac{1}{N} \sum_{i<j} \left[\beta^2 J^2 \sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + \frac{1}{2} \beta^2 J^2 n + \beta J_0 \sum_{\alpha} S_i^\alpha S_j^\alpha \right] + \beta h \sum_{i=1}^{N} \sum_{\alpha=1}^{n} S_i^\alpha \right\}$$ (6d)

$$= \exp\left[\frac{(N-1)\beta^2 J^2 n}{4} \right] \mathrm{Tr} \, \exp\left\{ \frac{1}{N} \sum_{i<j} \left[\beta^2 J^2 \sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + \beta J_0 \sum_{\alpha} S_i^\alpha S_j^\alpha \right] + \beta h \sum_{i=1}^{N} \sum_{\alpha=1}^{n} S_i^\alpha \right\},$$ (6e)

where the integral term in Eq. (6a) is calculated as

$$\mathcal{I} = \int dJ_{ij} \, \exp\left[-\frac{1}{2} \frac{N}{J^2} J_{ij}^2 + \left(\frac{J_0}{J^2} + \beta \sum_{\alpha=1}^{n} S_i^\alpha S_j^\alpha \right) J_{ij} \right]$$ (7a)

$$= \sqrt{\frac{4\pi J^2}{N}} \exp\left[\left(\frac{J_0^2}{J^4} + \frac{2J_0}{J^2} \beta \sum_{\alpha=1}^{n} S_i^\alpha S_j^\alpha + \beta^2 \sum_{\alpha,\beta}^{n} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta \right) \middle/ \frac{2N}{J^2} \right]$$ (7b)

$$= \sqrt{\frac{4\pi J^2}{N}} \exp\left(\frac{J_0^2}{2NJ^2} \right) \exp\left[\frac{1}{N} \left(\beta J_0 \sum_{\alpha=1}^{n} S_i^\alpha S_j^\alpha + \frac{1}{2} \beta^2 J^2 \sum_{\alpha,\beta}^{n} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta \right) \right],$$ (7c)

and the following trick is used in Eq. (6c)

$$\sum_{\alpha,\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta = 2 \sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + \sum_{\alpha} \left(S_i^\alpha S_j^\alpha \right)^2 = 2 \sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta + n.$$ (8)

Considering $\left(\sum_i A_i \right)^2 = \sum_i A_i^2 + \sum_{i\neq j} A_i A_j = \sum_i A_i^2 + 2 \sum_{i<j} A_i A_j$, i.e.

$$\sum_{i<j} A_i A_j = \frac{1}{2} \left[\left(\sum_i A_i \right)^2 - \sum_i A_i^2 \right],$$ (9)

we have

$$\frac{\beta^2 J^2}{N} \sum_{i<j} \sum_{\alpha<\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta = \frac{\beta^2 J^2}{2N} \left[\sum_{\alpha<\beta} \left(\sum_i S_i^\alpha S_i^\beta \right)^2 - \sum_i \sum_{\alpha<\beta} (S_i^\alpha)^2 (S_i^\beta)^2 \right] = \frac{\beta^2 J^2}{2N} \sum_{\alpha<\beta} \left(\sum_i S_i^\alpha S_i^\beta \right)^2 - \frac{\beta^2 J^2 n}{2},$$ (10)

and

$$\frac{\beta J_0}{N} \sum_{i<j} \sum_\alpha S_i^\alpha S_j^\alpha = \frac{\beta J_0}{2N} \left[\sum_\alpha \left(\sum_i S_i^\alpha \right)^2 - \sum_i \sum_\alpha (S_i^\alpha)^2 \right] = \frac{\beta J_0}{2N} \sum_\alpha \left(\sum_i S_i^\alpha \right)^2 - \frac{\beta J_0 n}{2}. \tag{11}$$

Thus Eq. (6e) is written as

$$\langle Z^n \rangle \propto \exp\left[\frac{N\beta^2 J^2 n}{4} \right] \mathrm{Tr} \, \exp\left[\frac{\beta^2 J^2}{2N} \sum_{\alpha<\beta} \left(\sum_i S_i^\alpha S_i^\beta \right)^2 + \frac{\beta J_0}{2N} \sum_\alpha \left(\sum_i S_i^\alpha \right)^2 + \beta h \sum_{i=1}^N \sum_{\alpha=1}^n S_i^\alpha \right], \tag{12}$$

where the following approximation in the large N limit is used

$$\exp\left[\frac{(N-3)\beta^2 J^2 n}{4} - \frac{\beta J_0 n}{2} \right] \approx \exp\left(\frac{N\beta^2 J^2 n}{4} \right). \tag{13}$$

In order to linearize the quadratic term on the exponential, it is useful to introduce the Hubbard-Stratonovich transform, an inverse application of the Gaussian integral, as follow

$$\exp\left(\frac{y^2}{2} \right) = \int_{-\infty}^\infty \frac{dx}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2} \right) \exp(xy). \tag{14}$$

Let $x = \beta J \sqrt{N} q_{\alpha\beta}$, and $y = \beta J \sum_i S_i^\alpha S_i^\beta / \sqrt{N}$, then we have

$$\exp \frac{\beta^2 J^2}{2N} \left(\sum_i S_i^\alpha S_i^\beta \right)^2 = \beta J \sqrt{N} \int_{-\infty}^\infty \frac{dq_{\alpha\beta}}{\sqrt{2\pi}} \exp\left(-\beta^2 J^2 N \frac{q_{\alpha\beta}^2}{2} + \beta^2 J^2 q_{\alpha\beta} \sum_i S_i^\alpha S_i^\beta \right). \tag{15}$$

Let $x = \sqrt{\beta J N} m_\alpha$, and $y = \sqrt{\beta J / N} \sum_i S_i^\alpha$, then we have

$$\exp \frac{\beta J_0}{2N} \left(\sum_i S_i^\alpha \right)^2 = \sqrt{\beta J N} \int_{-\infty}^\infty \frac{dm_\alpha}{\sqrt{2\pi}} \exp\left(-\beta J_0 N m_\alpha^2 + \beta J_0 m_\alpha \sum_i S_i^\alpha \right). \tag{16}$$

Then Eq. (12) can be written as

$$\langle Z^n \rangle \propto \exp\left[\frac{N\beta^2 J^2 n}{4} \right] \int_{-\infty}^\infty \prod_{\alpha<\beta} dq_{\alpha\beta} \prod_\alpha dm_\alpha \, \mathrm{Tr} \, \exp\left[-\frac{\beta^2 J^2 N}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{\beta J_0 N}{2} \sum_\alpha m_\alpha^2 \right]$$

$$\times \exp\left[\beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} \sum_i S_i^\alpha S_i^\beta + \beta J_0 \sum_\alpha m_\alpha \sum_i S_i^\alpha \right] \exp\left[\beta h \sum_{i=1}^N \sum_{\alpha=1}^n S_i^\alpha \right] \tag{17a}$$

$$= \exp\left[\frac{N\beta^2 J^2 n}{4} \right] \int_{-\infty}^\infty \prod_{\alpha<\beta} dq_{\alpha\beta} \prod_\alpha dm_\alpha \, \exp\left[-\frac{\beta^2 J^2 N}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{\beta J_0 N}{2} \sum_\alpha m_\alpha^2 \right]$$

$$\times \mathrm{Tr} \, \exp\left[\beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} \sum_i S_i^\alpha S_i^\beta + \beta \sum_\alpha (J_0 m_\alpha + h) \sum_i S_i^\alpha \right] \tag{17b}$$

$$= \exp\left[\frac{N\beta^2 J^2 n}{4} \right] \int_{-\infty}^\infty \prod_{\alpha<\beta} dq_{\alpha\beta} \prod_\alpha dm_\alpha \exp\left[N \left(-\frac{\beta^2 J^2}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{\beta J_0}{2} \sum_\alpha m_\alpha^2 + \log \mathrm{Tr} \, e^{\mathcal{L}} \right) \right], \tag{17c}$$

where we define

$$\mathcal{L} \equiv \beta^2 J^2 \sum_{\alpha<\beta} q_{\alpha\beta} S_i^\alpha S_i^\beta + \beta \sum_\alpha (J_0 m_\alpha + h) S_i^\alpha, \tag{18}$$

in Eq. (17b) and used

$$\prod_{i=1}^{N} \text{Tr } e^{\mathcal{L}} = \left(\text{Tr } e^{\mathcal{L}}\right)^{N} = \exp\left[N\log\left(\text{Tr } e^{\mathcal{L}}\right)\right].$$

(19)

In large N limit, the integral in Eq. (17c) can be calculated with the Laplace approximation, also known as the saddle-point approximation, i.e.

$$\int dm \, e^{N\mathcal{F}(m)} \xrightarrow{N\to\infty} e^{N\mathcal{F}_{\max}(m^*)}.$$

(20)

Let

$$\mathcal{F} = -\frac{\beta^2 J^2}{2}\sum_{\alpha<\beta} q_{\alpha\beta}^2 - \frac{\beta J_0}{2}\sum_{\alpha} m_{\alpha}^2 + \log\text{Tr } e^{\mathcal{L}},$$

(21)

and the result of the integral is

$$\langle Z^n\rangle \propto \exp\left[\frac{N\beta^2 J^2 n}{4} - \frac{\beta^2 J^2 N}{2}\sum_{\alpha<\beta}\left(q_{\alpha\beta}^{\star}\right)^2 - \frac{\beta J_0 N}{2}\sum_{\alpha}\left(m_{\alpha}^{\star}\right)^2 + N\log\text{Tr } e^{\mathcal{L}}\right]$$

(22a)

$$= \exp\left\{Nn\left[\frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2}{2n}\sum_{\alpha<\beta}\left(q_{\alpha\beta}^{\star}\right)^2 - \frac{\beta J_0}{2n}\sum_{\alpha}\left(m_{\alpha}^{\star}\right)^2 + \frac{1}{n}\log\text{Tr } e^{\mathcal{L}}\right]\right\}$$

(22b)

$$\approx 1 + Nn\left\{\frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2}{2n}\sum_{\alpha<\beta}\left(q_{\alpha\beta}^{\star}\right)^2 - \frac{\beta J_0}{2n}\sum_{\alpha}\left(m_{\alpha}^{\star}\right)^2 + \frac{1}{n}\log\text{Tr } e^{\mathcal{L}}\right\}$$

(22c)

where we used Taylor expansion in Eq. (22c), and $q_{\alpha\beta}^{\star}, m_{\alpha}^{\star} = \arg\max_{\{q_{\alpha\beta}, m_{\alpha}\}}\mathcal{F}$. Through $\frac{\partial}{\partial q_{\alpha\beta}}\mathcal{F} = \frac{\partial}{\partial m_{\alpha}}\mathcal{F} = 0$, we arrive at

$$q_{\alpha\beta}^{\star} = \frac{1}{\beta^2 J^2}\frac{\partial}{\partial q_{\alpha\beta}}\log\text{Tr } e^{\mathcal{L}} = \frac{1}{\beta^2 J^2}\frac{\text{Tr } e^{\mathcal{L}}\beta^2 J^2}{\text{Tr } e^{\mathcal{L}}}S^{\alpha}S^{\beta} = \frac{\text{Tr } S^{\alpha}S^{\beta}e^{\mathcal{L}}}{\text{Tr } e^{\mathcal{L}}},$$

(23)

$$m_{\alpha}^{\star} = \frac{1}{\beta J_0}\frac{\partial}{\partial m_{\alpha}}\log\text{Tr } e^{\mathcal{L}} = \frac{1}{\beta J_0}\frac{\text{Tr } e^{\mathcal{L}}\beta J_0}{\text{Tr } e^{\mathcal{L}}}S^{\alpha} = \frac{\text{Tr } S^{\alpha}e^{\mathcal{L}}}{\text{Tr } e^{\mathcal{L}}}.$$

(24)

The free energy density f is finally written as

$$f = -\frac{1}{N}\langle F\rangle = -\frac{1}{\beta}\lim_{n\to 0}\frac{\langle Z^n\rangle - 1}{Nn} = -\frac{1}{\beta}\lim_{n\to 0}\left\{\frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2}{2n}\sum_{\alpha<\beta}\left(q_{\alpha\beta}^{\star}\right)^2 - \frac{\beta J_0}{2n}\sum_{\alpha}\left(m_{\alpha}^{\star}\right)^2 + \frac{1}{n}\log\text{Tr } e^{\mathcal{L}}\right\}.$$

(25)

3 Replica Symmetry Ansatz

To continue solving Eq. (25), we need to consider the dependencies of $q_{\alpha\beta}$ and m^{α} for different replica index. A naive idea is that they are independent of index, i.e. $\forall\alpha,\beta, q_{\alpha\beta} = q, m_{\alpha} = m$, also called *replica symmetry ansatz*. The replica symmetric free energy is written as

$$f_{\text{RS}} = -\frac{1}{\beta}\lim_{n\to 0}\left\{\frac{\beta^2 J^2}{4} - \frac{\beta^2 J^2(n-1)}{4}q^2 - \frac{\beta J_0}{2}m^2 + \frac{1}{n}\log\text{Tr } e^{\mathcal{L}^*}\right\}$$

(26a)

$$= -\frac{1}{\beta}\left\{\frac{\beta^2 J^2}{4}\left(1 + q^2\right) - \frac{\beta J_0}{2}m^2 + \lim_{n\to 0}\frac{1}{n}\log\text{Tr } e^{\mathcal{L}^*}\right\},$$

(26b)

where $\mathcal{L}^{\star} \equiv \beta^2 J^2 q\sum_{\alpha<\beta}S^{\alpha}S^{\beta} + \beta\left(J_0 m + h\right)\sum_{\alpha}S^{\alpha}$.

The final item is calculated as

$$\frac{1}{n}\log\operatorname{Tr} e^{\mathcal{L}^*} = \frac{1}{n}\log\operatorname{Tr}\,\exp\left\{\frac{1}{2}\beta^2 J^2 q\left(\sum_\alpha S^\alpha\right)^2 - \frac{1}{2}\beta^2 J^2 q n + \beta\left(J_0 m + h\right)\sum_\alpha S^\alpha\right\} \tag{27a}$$

$$= \frac{1}{n}\log\left\{\exp\left(-\frac{\beta^2 J^2 q n}{2}\right)\operatorname{Tr}\,\exp\left[\frac{1}{2}\beta^2 J^2 q\left(\sum_\alpha S^\alpha\right)^2 + \beta\left(J_0 m + h\right)\sum_\alpha S^\alpha\right]\right\} \tag{27b}$$

$$= \frac{1}{n}\log\left\{\exp\left(-\frac{\beta^2 J^2 q n}{2}\right)\operatorname{Tr}\int Dz\,\exp\left(\beta J\sqrt{q}z\sum_\alpha S^\alpha + \beta\left(J_0 m + h\right)\sum_\alpha S^\alpha\right)\right\} \tag{27c}$$

$$= \frac{1}{n}\log\int Dz\,\exp\left\{n\log\left[2\cosh\left(\beta\hat{H}(z)\right)\right] - \frac{n}{2}\beta^2 J^2 q\right\} \tag{27d}$$

$$\approx \frac{1}{n}\log\left\{1 + n\int Dz\,\log\left[2\cosh\left(\beta\hat{H}(z)\right)\right] - \frac{n}{2}\beta^2 J^2 q\int Dz\right\} \tag{27e}$$

$$\approx \int Dz\,\log\left[2\cosh\left(\beta\hat{H}(z)\right)\right] - \frac{1}{2}\beta^2 J^2 q, \tag{27f}$$

where we used Hubbard-Stratonovich transform again in Eq. (27b) and reparameterized $\hat{z}$ by a standard Gaussian variable z, rewriting integral variables as Gaussian integral measures

$$\exp\left[\frac{1}{2}\beta^2 J^2 q\left(\sum_\alpha S^\alpha\right)^2\right] = \int d\hat{z}\sqrt{\frac{\beta^2 J^2 q}{2\pi}}\exp\left(-\frac{\hat{z}^2}{2}\beta^2 J^2 q\right)\exp\left(\beta^2 J^2 q\hat{z}\sum_\alpha S^\alpha\right) \tag{28a}$$

$$= \int \frac{dz}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\exp\left(\beta J\sqrt{q}z\sum_\alpha S^\alpha\right) \tag{28b}$$

$$= \int Dz\,\exp\left(\beta J\sqrt{q}z\sum_\alpha S^\alpha\right). \tag{28c}$$

The last item in Eq. (27c) is calculated as

$$\operatorname{Tr}\int Dz\,\exp\left(\beta J\sqrt{q}z\sum_\alpha S^\alpha + \beta\left(J_0 m + h\right)\sum_\alpha S^\alpha\right) = \int Dz\,\operatorname{Tr}\,\exp\left[\sum_\alpha S^\alpha\left(\beta J\sqrt{q}z + \beta\left(J_0 m + h\right)\right)\right] \tag{29a}$$

$$= \int Dz\prod_{\alpha=1}^{n}\operatorname{Tr}\,\exp\left[S^\alpha\beta\hat{H}(z)\right] \tag{29b}$$

$$= \int Dz\left\{2\cosh\left(\beta\hat{H}(z)\right)\right\}^n \tag{29c}$$

$$= \int Dz\,\exp\left\{n\log\left[2\cosh\left(\beta\hat{H}(z)\right)\right]\right\}, \tag{29d}$$

where we defined $\hat{H}(z) \equiv J\sqrt{q}z + (J_0 m + h)$.

Finally, the replica symmetric free energy is

$$f_{\rm RS} = -\frac{1}{\beta}\left\{\frac{\beta^2 J^2}{4}\left(1 + q^2\right) - \frac{\beta J_0}{2}m^2 + \lim_{n\to 0}\left\{\int Dz\,\log\left[2\cosh\left(\beta\hat{H}(z)\right)\right] - \frac{1}{2}\beta^2 J^2 q\right\}\right\} \tag{30a}$$

$$= \frac{\beta J^2}{4}(q-1)^2 + \frac{J_0}{2}m^2 - \frac{1}{\beta}\int Dz\,\log\left[2\cosh\left(\beta\hat{H}(z)\right)\right]. \tag{30b}$$

Through

$$\frac{\partial}{\partial m}f_{\rm RS} = -\beta J_0 m + \int Dz\,(\tanh\beta\hat{H}(z))\cdot\beta J_0 = 0, \tag{31}$$

$$\frac{\partial}{\partial q}f_{\rm RS} = \frac{\beta^2 J^2}{2}(q-1) + \int Dz\,(\tanh\beta\hat{H}(z))\cdot\frac{\beta J}{2\sqrt{q}}z = 0, \tag{32}$$

we obtain a set of closed equations, called saddle point equations

$$m = \int Dz \tanh \beta \hat{H}(z),$$

(33)

$$q = 1 - \int Dz \operatorname{sech}^2 \beta \hat{H}(z) = \int Dz \tanh^2 \beta \hat{H}(z).$$

(34)

4 Phase diagram

Considering a simple case where $h = 0$, we use numerical methods to iterate Eq. (33) and Eq. (34), and then calculate the free energy Eq. (30b) with the fixed points of m and q. The results of order parameters m and q are shown in Fig. 1, which (especially the interaction steps) recover the well-known phase diagram of the SK model as shown in Fig. 2(a) [4]. The free energy density with different J_0 and T is shown in Fig. 2(b).

Due to the Frustration, the spin in the SK model is frozen at low temperature, yet remains highly disordered, with the order parameter $m = 0$. But this is a phase different from the paramagnetic phase (also $m = 0$) and is called the spin glass phase. In short, $m \neq 0$ identifies the ferromagnetic phase, and the EA order parameter q is introduced to distinguish between the paramagnetic phase ($q = 0$) and the spin glass phase ($q \neq 0$).
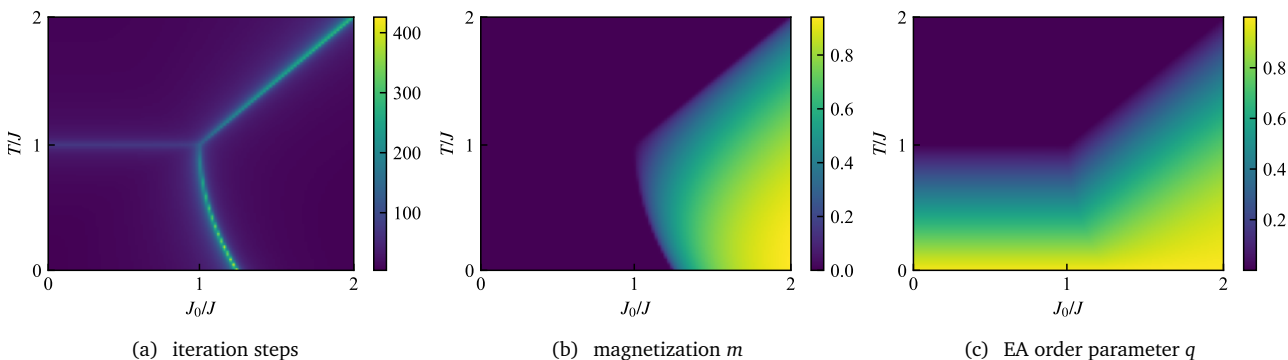

(a) iteration steps (b) magnetization m (c) EA order parameter q

Figure 1: The results of the replica symmetric solution for the SK model by numerical iteration.

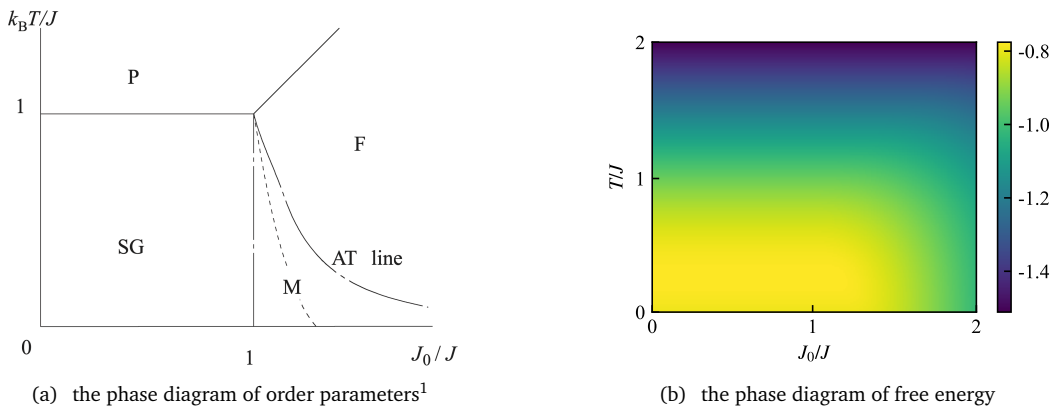(a) the phase diagram of order parameters[1] (b) the phase diagram of free energy

Figure 2: The phase diagram of the SK model.

[4] Nishimori, Hidetoshi, *Statistical Physics of Spin Glasses and Information Processing: An Introduction* (Oxford, 2001), p. 20